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STATIONARY POINTS AND FINITE-DIFFERENCE SCHEMES FOR DIFFERENTIA--ETC
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STATIONARY POINTS AND FINITE-DIFFERENCE SCHEMES FOR DIFFERENTIAL INCLUSIONS

Jean-Pierre Aubin

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Mathematics Research Center
University of Wisconsin—Madison
610 Walnut Street
Madison, Wisconsin 53706

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ABSTRACT

We present results of Aubin-Cellina-Nohel dealing with the properties of the differential inclusion

$$x' \in F(x), \quad x(0) = x_0$$

when F is an upper semicontinuous map from a compact convex set $K \subset \mathbb{R}^n$ to the compact convex subsets of $\mathbb{R}^n$. We prove that the tangential condition

$$\forall x \in K, \quad F(x) \cap T_K(x) \neq \emptyset$$

where $T_K(x)$ is the tangent cone to K at x implies

a) the existence of a stationary point $x_* \in K$, i.e., a solution to the inclusion $0 \in F(x)$,

b) the existence of a solution to the implicit finite difference scheme, i.e., a sequence of elements $x_n \in K$ satisfying

$$\forall n \geq 0, \quad x^{n+1} - x^n \in F(x^{n+1}), \quad x^0 = x_0 .$$

We recall that this tangential condition is necessary and sufficient for the existence of trajectories $x(\cdot)$ of the differential inclusion to satisfy $x(t) \in K$ for all $t \geq 0$.

AMS(MOS) Subject Classification: 47M10, 49B30

Key Words: Differential inclusions, stationary points, fixed points, invariant sets, implicit finite difference schemes

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SIGNIFICANCE AND EXPLANATION

So called "differential inclusions", in which the rate of change $\dot{x}$ of the state variable x is restricted to lie in a specified set $F(x)$ (and not given exactly) arise naturally in economics, control theory and other fields. As with differential equations, one is interested in the existence of stationary points. Here it is shown that if F is tangent to a compact convex set (in a sense made precise), then the differential inclusion has a stationary point within this set. Moreover, a discrete implicit difference scheme which approximates the inclusion will also be solvable when the tangency condition holds and the initial data is chosen from the set.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

STATIONARY POINTS AND FINITE-DIFFERENCE SCHEMES

FOR DIFFERENTIAL INCLUSIONS

Jean-Pierre Aubin

Introduction

We devote this paper to the study of conditions insuring the existence of stationary points and approximation of viable trajectories by solutions to implicit finite-difference schemes.

Let $K \subset X$ be compact convex, F be a set-valued map from K to X. We say that a state $x_* \in K$ is stationary if $0 \in F(x_*)$; in other words, the constant trajectory $t \mapsto x_*$ is a solution to the differential inclusion $x'(t) \in F(x(t))$.

We say that a solution $x(\cdot)$ to the differential inclusion $x'(t) \in F(x(t))$ is viable if

(1) $\forall t \geq 0 , \quad x(t) \in K .$

We recall Haddad's theorem: If F is a bounded upper semicontinuous map with compact convex values, a necessary and sufficient condition for the differential inclusion to have viable trajectories is that the tangential condition

(2) $\forall x \in K , \quad F(x) \cap T_K(x) \neq \emptyset$

holds true. Indeed, $T_K(x) \doteq c\ell(\bigcup_{\lambda \geq 0} \lambda(K-x))$ coincides with the Bouligand contingent cone $D_K(x)$ to K at x.

This condition means that for any state $x \in K$, one can find a velocity $v \in F(x)$ that is tangent to K at x, and thus, that "pushes" the trajectory back into K. Note that this tangential condition operates only at boundary points $x \in \partial K$, since $T_K(x) = X$ when $x \in \text{Int } K$.

When K is convex compact, this tangential condition implies also that there exist stationary points $x_* \in K$.

Approximation theory of solutions of differential equations or inclusions uses finite difference schemes. The simplest is the explicit finite difference scheme: Let $k \in \mathbb{N}$ be fixed. If the state $x^n \in K$ is known, we take $x^{n+1} \in K$ to be the solution to

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$$k(x^{n+1} - x^n) \in F(x^n) \quad (\text{or} \quad x^{n+1} \in (1 + kF)(x^n)) .$$

There is no reason why we can find x^{n+1} in K. So, we are led to study the implicit finite difference scheme: If $x^n \in K$ is know, find $x^{n+1} \in K$, a solution to

$$k(x^{n+1} - x^n) \in F(x^{n+1}) \qquad (\text{or} \quad x^{n+1} \in (1 - \frac{1}{k} F)^{-1}(x^n)) .$$

The second important result is that when K is convex and compact, the tangential condition implies that there exist solutions to the implicit finite difference scheme that are viable, i.e., satisfying $x^n \in K$ for all n. In other words, the tangential condition implies that $(1 - \frac{1}{k} F)^{-1}$ leaves K "invariant" (in the sense that $\forall x \in K$, $(1 - \frac{1}{k} F)^{-1}(x) \cap K \neq \emptyset$) whereas $1 + kF$ does not have necessarily this property.

We can compare this result with the ones concerning the case where $F \doteq -A$, where A is a maximal monotone map. We know that $F \doteq -A$ does satisfy the tangential condition on $K \doteq D(A)$ (see Brézis [1]) and the resolvants $(1 + \frac{1}{k} A)^{-1}$ are well defined (single-valued) maps (Minty's Theorem).

For proving these two results, we need the Brouwer fixed point theorem. Actually, we shall not use it, but an equivalent theorem, called the Ky Fan inequality (See Ky Fan [1] or Aubin [1] chapter 5, §6, p. 199-203). Although equivalent to the Brouwer fixed point theorem, Ky Fan's inequality is much more easy to use. We shall not prove this inequality in this paper.

The next question that arises is whether all trajectories of the differential inclusion $x'(t) \in F(x(t))$, $x(0) = x_0 \in K$ are viable: In this case, one says that K is invariant by F. We show that this is the case when F satisfies the strong tangential condition

(3) $\qquad \forall x \in K , \quad F(x) \subset T_K(x) .$

Now, we note that if two set-valued maps F_1 and F_2 satisfy the tangential condition (or the strong one), then $\alpha_1 F_1 + \alpha_2 F_2$ $(\alpha_1, \alpha_2 \geq 0)$ has the same property. Therefore, if the differential inclusions $x'(t) \in F_i(x(t))$ $(i=1,2)$ have viable trajectories) (and hence, stationary trajectories), so does the differential inclusion

$$x'(t) = \alpha_1 F_1(x(t)) + \alpha_2 F_2(x(t)) .$$

We also consider the stability problem: If F satisfies the tangential condition, do the set-valued maps G "close to F" also satisfy the tangential conditions? The answer is roughly this: If K has a nonempty interior, if F satisfies

(4) $\forall\, x \in K\,,\quad F(x) \subset \operatorname{Int} T_K(x)\,,$

(and is upper semicontinuous with compact values), then any set-valued map G satisfying

(5) $\forall\, x \in K\,,\quad G(x) \subset F(x) + \alpha B$ for some $\alpha > 0$

satisfies also the strong tangential condition: Therefore, if G is upper semicontinuous with compact convex values, G has also stationary points and leaves K invariant.

We continue with an application to control problems. Let U be a set of controls. A feedback control $\tilde{u}$ is a continuous map associating to every state $x \in K$ of the system a suitable control $\tilde{u}(x) \in U$. Let the map f from $K \times U$ to X describe the dynamics of the system. We want to find feedback controls $\tilde{u}$ such that the trajectories of the differential equation

(6) $x'(t) = f(x(t), \tilde{u}(x(t)))\,,\ x(0) = x_0$

are viable. We solve this problem when f is affine with respect to the controls and satisfy other suitable conditions.

We also consider m convex lower semicontinuous convex functions V_j defined on K and a convex lower semicontinuous convex function W defined on X. We shall prove that if

$$\forall\, x \in K\,,\quad \exists\, v \in F(x) \cap T_K(x) \text{ such that}$$

$$\max_{j=1,\ldots,m} DV_j(x)(v) + W(v) \neq \emptyset\,,$$

then there exists solutions $\{x^n\}_n$ to the implicit finite difference scheme that satisfy

$$\forall\, j=1,\ldots,m,\quad V_j(x^{n+1}) - V_j(x^n) + W(x^{n+1} - x^n) \leq 0\,.$$

When $W(v) = \|v\|$, such solutions converge to a stationary point x_* of F when $n \to \infty$.

Outline

1. Statement of the main theorem and applications
2. Ky Fan's inequality
3. Proof of existence of stationary points
4. Invariant subsets
5. Stability under perturbations
6. Feedback controls yielding viable trajectories
7. Lyapunov functions for implicit finite difference schemes.

1. Statement of the main theorem and applications

When K is a closed convex subset of X, the tangent cone $T_K(x) \doteq \mathrm{cl}\left(\bigcup_{h>0} h(K-x)\right)$ to K at x is convex: It is the set of elements $v \in X$ satisfying the condition

$$\liminf_{h \to 0+} \frac{d_K(x+hv)}{h} = 0$$

since $T_K(x)$ coincides with the contingent cone $D_K(x)$ [See Theorem 1.1 of Aubin [4]). Let us recall Haddad's theorem (see Haddad [1]) which holds true even when K is not convex provided that $T_K(x)$ is replaced by the contingent cone).

Theorem (Haddad)

Let K be a compact convex subset and F be a proper upper semicontinuous map with compact convex images from K to X. Assume that

(1) $$\forall x \in K, \quad F(x) \cap T_K(x) \neq \emptyset .$$

Then for all $x_0 \in K$, there exist solutions $x(\cdot) \in C(0,\infty;X)$ to the differential inclusion

(2) $$x'(t) \in F(x(t)), \quad x(0) = x_0$$

such that $x(t) \in K$ for all $t \geq 0$. ∎

We also recall that if X is finite dimensional and if there exists a viable trajectory for each initial state $x_0 \in K$, then the tangential condition (1) is necessary. It is remarkable that it is actually sufficient for the existence of stationary points as well as solutions to the implicit finite difference scheme.

Theorem 1

Let $K \subset X$ be compact convex and F be an upper hemicontinuous set-valued map from K into X with nonempty closed convex images. We posit the following tangential condition:

(1) $$\forall x \in K, \quad F(x) \cap T_K(x) \neq \emptyset .$$

Then

a) There exists a stationary point $x_* \in K$.

b) $\forall k \in \mathbb{N}$, $\forall x_0 \in K$, there exists a sequence of elements $x^n \in K$ such that $x^0 = x_0$ and

(3) $$\forall n \in \mathbb{N}, \; k(x^n - x^{n-1}) \in F(x^n) .$$ ∎

Remark. Statement b) is equivalent to

(4) $\forall\, y \in K$, there exists $x \in K$ such that $x-y \in \frac{1}{k} F(x)$

or, also, to

(5) $\forall\, y \in K\,,\quad (1 - \frac{1}{k} F)^{-1}(y) \cap K \neq \emptyset\,.$

Remark. Sum of two maps satisfying the tangential condition.

We single out a simple remark, although quite important, due to the convexity of the tangent cone. If two set-valued maps F_1 and F_2 satisfy the tangential condition, so does the set-valued map $\alpha_1 F_1 + \alpha_2 F_2$ when $\alpha_1, \alpha_2 \geq 0$.

Corollary 1

Let $K \subset X$ be convex compact and F_i $(i=1,2)$ be upper hemicontinuous maps from K to X with compact convex values. If

(6) $\forall\, i=1,2,\quad \forall\, x \in K\,,\quad F_i(x) \cap T_K(x) \neq \emptyset\,,$

then the differential inclusions

(7) $x' \in F_i(x) \quad (i=1,2) \quad \text{and} \quad x' \in \alpha_1 F_1(x) + \alpha_2 F_2(x)$

have both stationary points and viable solutions to the implicit finite difference scheme. ■

It is interesting to compare this situation with the case of maximal monotone maps: The sum of two maximal monotone maps is not necessarily maximal monotone.

Remark. An approximation theorem.

Theorem 1 provides actually an approximation theorem of solutions to differential inclusions.

Theorem 2

We posit the assumptions of Theorem 1 and we assume that F is bounded. Let $x_k(\cdot)$ be the piecewise-linear function defined on each interval $[n/k, (n+1)/k]$ by

(8) $x_k(t) \doteq x^n + (tk-n)(x^{n+1} - x^n)$

that interpolates a solution to the implicit finite difference scheme (3). Then a subsequence converges uniformly over every compact interval to a viable trajectory of the differential inclusion. ■

Proof.

It follows from the Convergence Theorem. See Aubin-Cellina [1] for instance. We know that

i) for all $t \geq 0$, $x_k(t) \in co(K)$, which is compact

ii) for all $t \geq 0$, $\|x_k'(t)\| \leq \sup_{x \in K} \sup_{v \in F(x)} \|v\| \doteq c < +\infty$.

For any $t \in [n/k,(n+1/k]$, we can write

$$(x_k(t),x_k'(t)) = (x^{n+1},k(x^{n+1} - x^n)) + (k(x^{n+1} - x^n)(t - (n+1/k),0) .$$

Since $\{x^n\}_n$ is a solution to the implicit finite difference scheme, $(x^{n+1},k(x^{n+1} - x^n)) \in \text{graph}(F)$ so that

iii) for all $t \geq 0$, $(x_k(t),x_k'(t)) \in \text{graph}(F) + \frac{c}{k}(B \times \{0\})$.

Hence the assumptions of the Convergence Theorem are satisfied and our theorem ensues.

■

Remark. Relaxation of the compactness assumption

We can replace the compactness assumption on K by a "coerciveness assumption" on F. For instance, we can prove the following result. We denote by

$$\sigma(F(x),p) \doteq \sup\{\langle p,v\rangle \mid v \in F(x)\}$$

the support function to $F(x)$.

Theorem 3

Let $K \subset \mathbb{R}^n$ be a closed convex subset and $F:K \to \mathbb{R}^n$ be an upper hemicontinuous set-valued map with closed convex values satisfying the "coerciveness assumption":

(9) $$\lim_{\substack{\|x\| \to \infty \\ x \in K}} \sigma(F(x),x) < 0 .$$

We posit the tangential condition

(10) $$\forall x \in K ,\ F(x) \cap T_K(x) \neq \emptyset .$$

Then there exists a stationary point $x_* \in K$ of F. If we posit the stronger coerciveness assumption

(11) $$\lim_{\substack{\|x\| \to \infty \\ x \in K}} \frac{\sigma(F(x),x)}{\|x\|} = -\infty$$

then, for all $x_0 \in K$, there exists a viable trajectory of the implicit finite difference scheme. ■

Proof.

It consists in showing that we can replace K by the compact subset $K \cap aB$ (for a large enough), which is a convex compact subset of $\mathbb{R}^n$. Indeed, there exists $\varepsilon > 0$ such that $\varepsilon < -\lim\limits_{\substack{\|x\| \to \infty \\ x \in K}} \sigma(F(x),x)$; consequently there exists $a > 0$ so large that $\|x\| \geq a$ implies that $\sigma(F(x),x) \leq 0$. We take the number a large enough so that $K \cap a \operatorname{Int} B \neq \emptyset$. Therefore, Proposition 4.1 of Aubin [4] implies that $F(x) \cap T_{aB}(x)$ and Theorem 5.1 of Aubin [4] that $T_{K \cap aB}(x) = T_K(x) \cap T_{aB}(x)$. So the tangential condition implies that

$$\forall\, x \in K \cap aB \ , \quad F(x) \cap T_{K \cap aB}(x) \neq \emptyset \ .$$

It suffices to apply Theorem 1.

For proving the second part of the theorem, we replace F by the map G defined by $G(x) \doteq \frac{1}{k} F(x) + y - x$, which satisfies the coerciveness assumption (9) whenever F satisfies the stronger coerciveness assumption (11). Then we can associate to any $y \in K$ a solution $x \in K$ to the inclusion $0 \in G(x)$, i.e., to the inclusion $k(x-y) \in F(x)$.

∎

Remark. Monotone trajectories of implicit schemes.

Let $K \subset X$ be a nonempty subset. We consider a preorder "$x \geq y$" on K, characterized by the set-valued map defined by

$$P(x) = \{y \in K \mid y \geq x\}.$$

This set-valued map satisfies

$$\forall\, x \in K \ , \quad x \in P(x) \quad \text{(reflexivity)}$$

$$\forall\, x \in K \ , \quad \forall\, y \in P(x), \quad P(y) \subset P(x) \quad \text{(transitivity)}.$$

If P is a set-valued map satisfying these properties, the associated preorder is defined by $y \geq x$ if and only if $y \in P(x)$. Let F be a set-valued map from K into X.

Theorem 4

Let $K \subset X$ be a nonempty compact convex subset, F be an upper hemicontinuous set-valued map from K to X with closed convex values. Let $P: K \to K$ be a set-valued map with closed convex values that is reflexive and transitive. We posit the following condition

(12) $\quad \forall x \in K, \quad F(x) \cap T_{P(x)}(x) \neq \emptyset .$

Then,

a) For any $x_0 \in K$, there exists a stationary point x_* of F such that $x_* \geq x_0$.

b) For all $k \in \mathbb{N}$, there exists a sequence of states $x^n \in K$ which is a solution to the implicit scheme

$$
(13) \quad \begin{cases} \forall n \geq 1, \quad \text{i)} \quad k(x^n - x^{n-1}) \in F(x^n) \quad \text{where } x^0 = x_0 \text{ is given.} \\ \qquad\qquad \text{ii)} \quad x^n \geq x^{n-1} . \end{cases}
$$

<u>Proof</u>.

We use <u>Theorem</u> 1 with $K = P(x_0) \subset X$. Indeed, for any $y \in P(x_0)$, we have $P(y) \subset P(x_0)$ and thus, $T_{P(y)}(y) \subset T_{P(x_0)}(y)$. Since $F(y) \cap T_{P(y)}(y) \neq \emptyset$, we deduce that $F(y) \cap T_{P(x_0)}(y) \neq \emptyset$ for all $y \in P(x_0)$. Hence it ensues that

a) there exists $x_* \in P(x_0)$ such that $0 \in F(x_*)$,

b) there exists $x^1 \in P(x_0)$ such that $k(x^1 - x^0) \in F(x^1)$.

By repeating the latter argument, we prove the existence of a solution to the implicit scheme (13). ∎

2. Ky Fan's inequality

Instead of using the Brouwer fixed point theorem, which states that any continuous map from a compact convex subset of $\mathbb{R}^n$ to itself has a fixed point, we shall use an equivalent statement, Ky Fan's inequality, equivalent but still much more powerful.

Theorem 1

Let K be a convex compact subset and $\varphi : K \times K \to \mathbb{R}$ be a function satisfying

$$
(1)\quad \begin{cases} \text{i)} & \forall y \in K, \quad x \mapsto \varphi(x,y) \text{ is lower semicontinuous} \\ \text{ii)} & \forall x \in K, \quad y \mapsto \varphi(x,y) \text{ is concave} \\ \text{iii)} & \forall y \in K, \quad \varphi(y,y) \le 0 . \end{cases}
$$

Then there exists $\bar{x} \in K$ such that

$$
(2)\qquad \forall y \in K, \quad \varphi(\bar{x},y) \le 0 . \qquad \blacksquare
$$

Proof.

a) Ky Fan's inequality implies Brouwer's fixed point theorem: Let K be a convex compact subset of $\mathbb{R}^n$ and f be a continuous map from K to K. Hence the function φ defined by $\varphi(x,y) = \langle f(x) - x, y - x \rangle$ satisfies Assumption (1) of Theorem 1: Hence there exists $\bar{x} \in X$ such that $\varphi(\bar{x},y) \le 0$ for all $y \in K$ and, in particular, such that $\| f(\bar{x}) - \bar{x} \| = \varphi(\bar{x}, f(\bar{x})) \le 0$. Therefore $\bar{x}$ is a fixed point of f.

b) Brouwer's fixed point theorem implies Ky Fan's inequality: See for instance, Aubin [1], Chapter 5, §6, p. 199-203 for a proof. For more sophisticated results, see Aubin [2], Chapter 7, §1 and Chapter 13, §2. $\blacksquare$

3. Proof of existence of stationary points.

Theorem 1.1 follows obviously from the following theorem.

Theorem 1

Let $K \subset X$ be compact convex and F be an upper hemicontinuous map from K to X with closed convex images. We posit the following tangential condition

(1) $$\forall x \in K\ ,\ F(x) \cap T_K(x) \neq \emptyset\ .$$

Then

a) there exists a stationary point $x_* \in K$ of F

b) $\forall y \in K\ ,\ \exists\ \hat{x} \in K$ such that $\hat{x} - y \in F(\hat{x})$. ■

The second condition amounts to saying that $(1 - F)^{-1} \cap K$ is a proper set-valued map from K to K (with nonempty images).

Proof.

a) The second conclusion follows from the first: Since the set-valued map G defined by $G(x) = F(x) + y - x$ is the sum of the two maps F and $y-1$ that satisfy the tangential condition, then G satisfies it and consequently, has a stationary point $\hat{x}$ that is a solution to the inclusion $\hat{x} - y \in F(\hat{x})$.

b) We denote by $\sigma(F(x),q) \doteq \sup_{v \in F(x)} \langle q,v \rangle$ the support function of the closed convex subset $F(x)$.

For proving the existence of a stationary point, we assume the contrary: $\forall x \in K$; $0 \notin F(x)$ and we derive a contradiction. Since the subsets $F(x)$ are closed and convex, the separation theorem implies that:

(3) $$\forall x \in K\ ,\ \exists\ p \in X^* \text{ such that } \sigma(F(x),-p) < 0\ .$$

We set

(4) $$\Delta_p = \{x \in K \mid \sigma(F(x),-P) < 0\}.$$

So, the statement that no stationary point exists takes the form

(5) $$K \subset \bigcup_{p \in X^*} \Delta_p\ .$$

c) Since F is upper hemicontinuous, the subsets Δ_p are open. Hence the compact subset K can be covered by n open subsets Δ_{p_i} . Let $\{a_i\}_{i=1,\dots,n}$ be a continuous partition of unity associated to this covering .

d) We introduce the function φ defined on $K \times K$ by

(6) $$\varphi(x,y) = -\sum_{i=1}^{n} a_i(x) \langle p_i, x-y \rangle .$$

It is continuous with respect to x, affine with respect to y and satisfies $\varphi(y,y) = 0$ for all $y \in K$.

So the assumptions of the Ky Fan inequality hold and, consequently, there exists $x_* \in K$ such that

(7) $$\forall y \in K, \quad \varphi(x_*,y) = \langle -p_*, x_* - y \rangle \leq 0$$

where we set $p_* = \sum_{i=1}^{n} a_i(x_*)p_i$.

In other words, p_* belongs to $N_K(x_*)$. The tangential condition (1) implies the existence of $v \in F(x_*) \cap T_K(x_*)$. Hence, since $N_K(x) = T_K(x)^-$,

(8) $$\sigma(F(x_*), -p_*) \geq \langle -p_*, v \rangle \geq 0 .$$

e) The latter inequality is impossible: Let I be the set of indices i such that $a_i(x_*) > 0$. It is nonempty since $\sum_{i=1}^{n} a_i(x_*) = 1$. If $i \in I$, then $x_* \in \Sigma_i$ and thus, $\sigma(F(x_*),-p_i) < 0$. Therefore:

$$\sigma(F(x_*),-p_*) = \sigma(F(x_*),-\sum_{i \in I} a_i(x_*)p_i)$$

$$\leq \sum_{i \in I} a_i(x_*)\sigma(F(x_*),-p_i) \text{ (by the convexity of support functions).}$$

We have proved that $\sigma(F(x_*),-p_*) < 0$, which is the contradiction we were looking for.

∎

Remark

We used only the weaker tangential condition

(9) $$\forall x \in K, \quad \forall p \in N_K(x), \quad \sigma(F(x),-p) \geq 0 .$$

Remark that condition (9) is equivalent to the tangential condition (1) when the images $F(x)$ are convex compact: If $F(x) \cap T_K(x) = \emptyset$, then $0 \notin F(x) - T_K(x)$, which is a closed convex set. The separation theorem implies the existence of $p \in X^*$ such that

$$\sigma(F(x),-p) + \sup_{v \in T_K(x)} \langle -p,-v \rangle \leq -\varepsilon < 0 .$$

Hence $\sigma(T_K(x),p)$ is bounded above and thus, is equal to 0 and $p \in T_K(x)^- = N_K(x)$. This is a contradiction of condition (1). ∎

Remark. The Kakutani fixed point theorem.

We deduce from Theorem 1 the Kakutani fixed point theorem.

Theorem 2. Kakutani

Let K be a compact convex subset and G be an upper semicontinuous map from K to K with compact convex images. Then there exists a fixed point $x_* \in K$ of G. ∎

Proof.

We set $F(x) \doteq G(x) - x \subset K - x \subset T_K(x)$. Hence $F(\cdot)$ is an upper hemicontinuous map from K to X that satisfies the tangential condition (1). By Theorem 1, it is a stationary point $x_* \in K$, which is a fixed point of G. ∎

Actually, the same proof implies the following statement. We say that G is inward if

$$(10) \qquad \forall x \in K, \quad G(x) \subset x + T_K(x) .$$

Theorem 3. Browder-Ky Fan

Let G be an upper hemicontinuous map from a compact convex subset $K \subset X$ to the closed convex subsets of X. If G is inward, then it has a fixed point $x_* \in K$. ∎

We also mention the following result.

We say that G is outward if

$$(11) \qquad \forall x \in K, \quad G(x) \subset x - T_K(x) .$$

Theorem 4. Ky Fan - Rogalski

Let G be an upper hemicontinuous map from a compact convex subset $K \subset X$ to the closed convex subsets of X. If G is outward, then

a) it has a fixed point $x_* \in K$

b) $K \subset G(K)$. (i.e., for all $y \in K$, $\exists\, x \in K$ such that $y \in G(x)$.) ∎

Proof. It follows from Theorem 1 applied to the map F defined on K by $F(x) \doteq x - G(x)$. ∎

Remark

It is not more difficult to solve a slightly more general theorem. Let X and Y be two Hilbert spaces (or, more generally, Hausdorff locally convex vector spaces).

Theorem

We introduce

$$
(12)\quad \begin{cases} \text{i)} & K, \text{ a compact convex subset of } X, \\ \text{ii)} & F, \text{ an upper hemicontinuous map from } K \text{ to } Y \text{ with closed convex values,} \\ \text{iii)} & A : K \to \mathcal{L}(X,Y), \text{ a continuous map associating with each } x \in K \text{ a continuous linear operator from } X \text{ to } Y. \end{cases}
$$

We posit the tangential condition.

$$
(13)\qquad \forall x \in K, \quad F(x) \cap c\ell(A(x)T_K(x)) \neq \emptyset
$$

holds, then

a) there exists a stationary point $x_* \in K$ of F: $0 \in F(x_*)$

b) $\forall y \in K$, there exists $\hat{x} \in K$ satisfying

$$
A(\hat{x})(\hat{x} - y) \in F(\hat{x}) . \qquad \blacksquare
$$

Note that the second statement allows the construction of a solution to the implicit finite difference scheme:

$$
A(x^n)(x^n - x^{n-1}) \in F(x^n); \quad x^0 = x_0 .
$$

It also amounts to saying that the set-valued map $x \mapsto (x - A(x)^{-1}F(x)) \cap K$ from K to K has nonempty values.

When $A \in \mathcal{L}(X,Y)$ does not depend on x, the second statement can be stated as follows.

$$
(14)\quad \begin{cases} \text{Whenever there exists a solution } x_0 \text{ in } K \text{ to the linear equation } y = A x_0, \\ \text{then there exists also a solution } \hat{x} \text{ in } K \text{ to the perturbed inclusion} \\ y \in A(\hat{x}) + F(\hat{x}) . \end{cases}
$$

Proof.

a) The second statement follows from the first applied to the map G defined by $G(x) = F(x) + A(x)(y - x)$.

b) The proof of the first statement is the same as the proof of Theorem 1, where the function φ defined by (6) is replaced by the function φ defined by

$$
(15)\qquad \varphi(x,y) = -\sum_{i=1}^{n} a_i(x) \langle p_i, A(x)(x - y) \rangle . \qquad \blacksquare
$$

4. Invariant subsets

Let $\Omega \subset X$ be a nonempty open subset and F be a set-valued map from Ω into X.

Definition 1

We shall say that a subset $K \subset \Omega$ is "invariant" by F if for all $x_0 \in K$, all the trajectories of the differential inclusion $x'(t) \in F(x(t))$, $x(0) = x_0$, remain in K. ∎

When K is a closed convex subset, we give a sufficient condition for invariance.

We recall that π_K denotes the projector of best approximation to K.

Proposition 1

Let us assume that a set-valued map F satisfies the "strong external tangential condition"

(1) $$\forall x \in \Omega, \quad \forall v \in F(x), \quad \langle x - \pi_K(x), v \rangle \leq 0 .$$

Then K is invariant by F. ∎

Proof.

Let us assume that the statement is false: There exists an absolutely continuous function $x(\cdot)$ satisfying $x'(t) \in F(x(t))$ a.e., $x(0) = x_0 \in K$ and, for some $t_1 \in]0,T]$, $x(t_1) \notin K$.

Let us consider the function $t \mapsto d_K^2(x(t))$.

It is differentiable and it is easy to show that $\nabla d_K^2(x) = 2(x - \pi_K(x))$. Since $x(\cdot)$ is absolutely continuous, we deduce that for almost all $t \in [0,T]$,

$$\frac{d}{dt} d_K^2(x(t)) = \langle \nabla d_K^2(x(t)), x'(t) \rangle = 2 \langle x(t) - \pi_K(x(t)), x'(t) \rangle .$$

If $x(t) \in K$, we obtain that $\frac{d}{dt} d_K^2(x(t)) = 0$. If $x(t) \notin K$, we deduce from the strong external tangential condition that $\frac{d}{dt} d_K^2(x(t)) \leq 0$ since $x'(t) \in F(x(t))$.

Hence

$$0 < d_K^2(x(t_1)) - d_K^2(x(0)) = \int_0^{t_1} \frac{d}{dt} d_K^2(x(\tau))d\tau \leq 0 .$$

We have obtained a contradiction. ∎

As a corollary, we obtain the following result on invariant subsets:

Theorem 1

Let K be a closed convex subset of $\mathbb{R}^n$ and $F:K \to \mathbb{R}^n$ be an upper semi-continuous set-valued map with compact convex images. If $F(K)$ is relatively compact and if the "strong tangential condition"

$$(2) \qquad \forall x \in K , \quad F(x) \subset T_K(x) ,$$

then, for any $x_0 \in K$, there exists a solution $x(\cdot)$ to the problem

$$(3) \quad \begin{cases} \text{i)} & \forall t \geq 0 , \quad x(t) \in K \quad (\text{and} \quad x(0) = x_0) \\ \text{ii)} & \forall \text{ a.a. } t \geq 0 , \quad x'(t) \in F(x(t)) \end{cases}$$

and K is invariant by F. ∎

Proof.

We associate to F its extension $G : \mathbb{R}^n \to \mathbb{R}^n$ defined by $G(x) \doteq F(\pi_K(x))$, which is upper semicontinuous with compact convex images, such that $G(\mathbb{R}^n) = F(K)$ is compact. Hence there exist solutions to the differential inclusion $x'(t) \in G(x(t)) = F(\pi_K(x(t)))$; $x(0) = x_0$. (See Aubin-Cellina [1].) But G satisfies the strong external tangential condition. Hence K is invariant by G thanks to the above proposition. Since $G(x) = F(x)$ for all $x \in K$, K is also invariant by F and the viable solutions to the differential inclusion for G are the viable solutions to the differential inclusion for F. ∎

5. Stability under perturbations

We assume now that $X = \mathbb{R}^n$ and that K is a compact convex subset whose interior is not empty. Hence we know that the graph of the set-valued map $y \to \operatorname{Int} T_K(y)$ is open (See Proposition 3.1 of Aubin [4].).

We prove now the following stability result:

Proposition 1

Let F be an upper semicontinuous compact valued map from K to X satisfying the "strong internal tangential condition":

$$(1) \qquad \forall x \in K, \quad F(x) \subset \operatorname{Int} T_K(x) .$$

Then there exists $\alpha > 0$ such that any set-valued map G "close" to F in the sense that

$$(2) \qquad \forall x \in K, \quad G(x) \subset F(x) + \alpha B$$

satisfies also the strong internal tangential condition. ■

Proof.

Since the graph of $\operatorname{Int} T_K(\cdot)$ is open, since the subsets $F(y)$ are compact and since F is u.s.c., we can associate with any neighborhood $N(y)$ of y on $\alpha_y > 0$ such that, for all $x \in N(y)$, we obtain

$$(3) \qquad \begin{cases} \text{i)} \ F(y) + 3\alpha_y B \subset T_K(x) \\ \text{ii)} \ F(x) \subset F(y) + \alpha_y B. \end{cases}$$

The compact set K can be covered by n neighborhoods $N(y_i)$. Let $\alpha \doteq \min \alpha_{y_i} > 0$. We deduce that

$$(4) \qquad \forall x \in K, \quad F(x) + 2\alpha B \subset T_K(x) .$$

Hence, if $G(x) \subset F(x) + \alpha B$ for all $x \in K$, we deduce that $G(x) + \alpha B \subset T_K(x)$ for all $x \in K$. ■

This yields a very important property of stability:

Theorem 1

Let $K \subset \mathbb{R}^n$ be a compact convex subset with nonempty interior and F be an upper semicontinuous map from K to $\mathbb{R}^n$ with convex compact values. Assume that it satisfies the strong internal tangential condition

(5) $\forall x \in K , \quad F(x) \subset \operatorname{Int} T_K(x) .$

Then there exists $\alpha > 0$ such that, every upper semicontinuous map from K to $\mathbb{R}^n$ with compact convex values satisfying

(6) $\forall x \in K , \quad G(x) \subset F(x) + \alpha B$

has stationary points in K and leaves K invariant. ■

6. Feedback controls yielding viable trajectories.

Let us consider a control problem

$$(1)\qquad \begin{cases} \text{i)} & x'(t) = f(x(t),u(t)) \\ \text{ii)} & x(0) = x_0 \end{cases}$$

where $x(t) \in K$ is the state of a dynamical system at time t and where $u(t) \in U$ is a control, which has to be chosen.

A "feedback control" is, by definition, a continuous (single-valued) map u from K to U. They are also called "closed loop" controls. If a feedback control is chosen, then the state evolves according to the differential equation

$$(2)\qquad \begin{cases} \text{i)} & x'(t) = f(x(t),\, \underset{\sim}{u}(x(t))) \\ \text{ii)} & x(0) = x_0 \in K\,. \end{cases}$$

The problem arises whether there exist feedback controls yielding a viable trajectory.

Theorem 1

Let us assume that $K \subset \mathbb{R}^n$ and U are both compact convex subsets and that $f: K \times U \to \mathbb{R}^n$ is a continuous map that is affine with respect to the controls u. We assume that there exists $\gamma > 0$ such that

$$(3)\qquad \begin{cases} \forall x \in K\,,\ \forall y \in \mathbb{R}^n,\ \|y\| \leq \gamma\,,\ \exists u \in U \text{ satisfying} \\ \quad f(x,u) - y \in T_K(x)\,. \end{cases}$$

Then there exists a feefback control $\underset{\sim}{u} \in \mathcal{C}(K,U)$ that yields viable trajectories as well as stationary trajectories x_* (solution of $f(x_*,\underset{\sim}{u}(x_*)) = 0$). ∎

Proof.

We have to prove the existence of a feedback control $\underset{\sim}{u} \in \mathcal{C}(K,U)$ such that

$$(4)\qquad \forall x \in K\,,\ f(x,\underset{\sim}{u}(x)) \in T_K(x)\,.$$

Indeed, Theorems 1.1 and 4.1 imply that the differential equation (2) has viable as well as stationary trajectories. Let us set

$$(5)\qquad C(x) \doteq \{u \in U \mid f(x,u) \in T_K(x)\}\,.$$

We note that assumption (3) implies that the subsets $C(x)$ are nonempty (take $y = x$). They are obviously compact and convex. (We recall that $y \mapsto T_K(y)$ is lower semicontinuous). Hence the set-valued map C is lower semicontinuous. (See Aubin-Cellina [1]).

Since K is compact, Michael's theorem implies the existence of a continuous selection $\underset{\sim}{u}$ of C, which is a feedback control we are looking for. ∎

7. Lyapunov functions for implicit finite difference schemes.

We consider now n functions V_j and we shall prove the existence of trajectories of the implicit finite difference scheme that satisfy inequalities of the form

$$\forall\, j=1,\dots,m,\quad \forall\, n \geq 0\ ,\quad V_j(x^{n+1}) - V_j(x^n) + W(x^{n+1} - x^n) \leq 0\ .$$

Theorem 1.

Let $K \subset X$ be compact convex, F be an upper hemicontinuous map from K to X with closed convex values. Let $V_j\,(j=1,\dots,m)$ and W be lower semicontinuous convex functions; we assume that the functions V_j are continuous at a same point of K. We posit the following assumption.

$$(1)\qquad \begin{cases} \forall\, x \in K\ , \quad \exists\, v \in F(x) \cap T_K(x) \text{ satisfying} \\ \forall\, j=1,\dots,m\ ,\quad DV_j(x)(v) + W(v) \leq 0\ . \end{cases}$$

Then, for any $x_0 \in K$, there exists a solution $\{x^n\}_{n\geq 0}$ of the implicit finite difference scheme

$$(2)\qquad k(x^{n+1} - x^n) \in F(x^{n+1}),\ x^{n+1} \in K\ ,\ x^0 = x_0$$

satisfying

$$(3)\qquad \max_{j=1,\dots,m} (V_j(x^{n+1}) - V_j(x^n)) + \frac{1}{k} W(k(x^{n+1} - x^n)) \leq 0 \qquad \blacksquare$$

Proof. For simplicity, we take $k = 1$.

We have to prove that

$$(4)\qquad \forall\, p \in X^*\ ,\ \langle p, x-x_0 \rangle \leq \sigma(F(x),p)$$

and that,

$$(5)\qquad \forall\, j=1,\dots,m\ ,\quad \forall\, q \in X^*\ ,\quad V_j(x) + \langle q, x-x_0 \rangle - W^*(q) \leq V_j(x_0)$$

since $W(x-x_0) = \sup\{\langle q, x-x_0 \rangle - W^*(q) \mid q \in X^*\}$.

Let us set

$$(6)\qquad \Delta_p = \{x \in K \mid \sigma(F(x),-p) + \langle p, x-x_0 \rangle < 0\}$$

and

$$(7)\qquad \Delta_q^j = \{x \in K \mid V_j(x) - V_j(x_0) + \langle q, x-x_0 \rangle - W^*(q) > 0\}\ .$$

Let us assume that the conclusion is false: We can express that by saying that

$$(8)\qquad K \subset \bigcup_{p\in X^*} \Delta_p \cup \bigcup_{j=1}^{m} \bigcup_{q\in X^*} \Delta_q^j\ .$$

Since F is upper hemicontinuous and since the functions V_j are lower semicontinuous, the subsets Δ_p and Δ^j_q are open. Hence the compact subset K can be covered by a family of ℓ subsets Δ_{p_i} and m_j subsets $\Delta^j_{q^j_k}$ $(j=1,\dots,n)$. Let us consider a continuous partition of unity $\{\{a_i\}_{i=1,\dots,\ell}, \{\{a^j_k\}_{k=1,\dots,m_j}\}_{j=1,\dots,n}\}$ associated to this covering of K.

We introduce the function φ defined on $K \times K$ by

(8) $$\varphi(x,y) = -\sum_{i=1}^{\ell} a_i(x) \langle p_i, x-y \rangle + \sum_{j=1}^{n} \sum_{k=1}^{m_j} a^j_k(x)(V_j(x)-V_j(y) + \langle q^j_k, x-y \rangle).$$

The functions $x \mapsto \varphi(x,y)$ are lower semicontinuous, $y \mapsto \varphi(x,y)$ are concave and $\varphi(y,y) = 0$ for all $y \in K$. So the assumptions of the Ky Fan inequality are satisfied (See Theorem 2.1).

Then there exists $x_* \in K$ such that

(9) $$\forall\, y \in K, \quad \varphi(x_*,y) \le 0 .$$

Let us set $\lambda^j_* = \sum_{k=1}^{m_j} a^j_k(x_*)$, $\lambda_* = \sum_{j=1}^{n} \lambda^j_*$, $q_* = 0$ if $\lambda_* = 0$ and $q_* = \frac{1}{\lambda_*} \sum_{j=1}^{m} \sum_{k=1}^{m_j} a^j_k(x_*) q^j_k$ if $\lambda_* > 0$, $p_* = 0$ if $\lambda_* = 1$ and $p_* = \frac{1}{1-\lambda_*} \sum_{i=1}^{\ell} a_i(x_*) p_i$ if $\lambda_* < 1$.

Inequalities (9) mean that x_* minimizes over K the function $x \mapsto \sum_{j=1}^{n} \lambda^j_* V_j(x) + \langle \lambda_* q_* - (1-\lambda_*)p_*, x \rangle$. Since the functions V_j are continuous at a same point $\bar{x} \in K$, we deduce that

(10) $$0 \in (1-\lambda_*)p_* + \lambda_* q_* + \sum_{j=1}^{m} \lambda^j_* \,\partial V_j(x_*) + N_K(x_*) .$$

We now use assumption (1): There exists $v_* \in F(x_*) \cap T_K(x_*)$ such that $DV_j(x_*)(v_*) + W(v_*) \le 0$ for all $j=1,\dots,n$. Then by taking the duality product of (10) with v_*, we obtain, by recalling that $DV_j(x_*)(v_*) = \sup(\langle q, v_* \rangle \mid q \in \partial V_j(x_*))$,

(11) $$\begin{cases} 0 \le (1-\lambda_*)\langle -p_*, v_* \rangle + \lambda_* \langle q_*, v_* \rangle - \lambda_* W(v_*) \\ \quad \le (1-\lambda_*)\sigma(F(x_*), -p_*) + \lambda_* W^*(q_*) . \end{cases}$$

We are able now to derive a contradiction of (9) by showing that

$$\varphi(x_*, x_0) > 0 .$$

Indeed, if $a_i(x_*) > 0$, then $x_* \in M_{p_i}$ and $\langle -p_i, x_* - x_0\rangle \geq \sigma(F(x_*),-p_i)$ and if $a_k^j(x_*) > 0$, then $x_* \in M^j_{q^j_k}$ and $V_j(x_*) - V_j(x_0) + \langle q^j_k, x_* - x_0\rangle \geq W^*(q^j_k)$. Thanks to the convexity of the functions $\sigma(F(x_*),\cdot)$ and $W^*(\cdot)$, we obtain

$$\phi(x_*,x_0) \geq (1 - \lambda_*)\sigma(F(x_*),-p_*) + \lambda_* W^*(q_*)$$

(since at least one of the numbers $a_i(x_*)$ or $a_k^j(x_*)$ is strictly positive). Hence (11) implies that $\phi(x_*,x_0) > 0$. ∎

<u>Example</u>

The main example is when $W(v) = \|v\|$. We obtain in this case the following example:

<u>Theorem</u> (Aubin-Siegel)

Let $K \subset X$ be compact convex, F be an upper hemicontinuous map from K to X with closed values. Let $V_j (j=1,\ldots,m)$ be m continuous convex functions that are continuous at a same point of K and bounded below. Assume that

(12) $$\forall x \in K, \quad F(x) \cap T_K(x) \cap \Big(\bigcup_{j=1}^{m} \partial V_j(x) + B\Big)^- \neq \emptyset .$$

Then, for any $x_0 \in K$, there exists a solution $\{x^n\}_n$ of the implicit finite difference scheme satisfying

(13) $$\forall\, n=1,\ldots,m, \quad \forall\, n \geq 0, \quad V_j(x^{n+1}) - V_j(x^n) + \|x^{n+1} - x^n\| \leq 0 .$$

Such trajectories converge to a stationary point of F. ∎

<u>*Proof*</u>.

Since $\|v\| = \sigma_B(v)$ is the support function of the unit ball, and since $DV_j(x)(v) = \sigma(\partial V_j(x),v)$ is the support function of $\partial V_j(x)$, then assumption (1) of Theorem 1 can be written in the form (12). Hence there exists a sequence of elements $x^n \in X$ satisfying (13). The sequences of real numbers $V_j(x_n)$ are decreasing and bounded below: Thus they converge to real numbers V_j. By summing up inequalities (13) from $n=p$ to $n=q-1$, we find that

$$\|x^q - x^p\| \leq \sum_{n=p}^{q-1} \|x^{n+1} - x^n\| \leq V_j(x^p) - V_j(x^q).$$

Since the right-hand side of this inequality converges to $v_j - v_j = 0$ when p and q go to ∞, we deduce that the sequence of elements $x^n \in K$ is a Cauchy sequence. Hence it converges to some $x_* \in K$. Since $x^{n+1} - x^n \in F(x^{n+1})$ and since the graph of F is closed, we deduce that $0 \in F(x_*)$. Hence x_* is a stationary point of F. ∎

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We present results of Aubin-Cellina-Nohel dealing with the properties of the differential inclusion $x' \in F(x)$, $x(0) = x_0$ when F is an upper semicontinuous map from a compact convex set $K \subset \mathbb{R}^n$ to the compact convex subsets of $\mathbb{R}^n$. We prove that the tangential condition $\forall\, x \in K, F(x) \cap T_K(x) \neq \emptyset$ where $T_K(x)$ is the tangent cone to K at x implies a) the existence of a stationary point $x_* \in K$, i.e., a solution to the inclusion $0 \in F(x)$, b) the existence of a solution to the implicit finite difference scheme, i.e., a sequence of elements $x_n \in K$ satisfying $\forall\, n \geq 0$, $x^{n+1} - x^n \in F(x^{n+1})$, $x^0 = x_0$.

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Abstract (continued)

We recall that this tangential condition is necessary and sufficient for the existence of trajectories $x(\cdot)$ of the differential inclusion to satisfy $x(t) \in K$ for all $t \geq 0$.